

CANNON–THURSTON MAPS DO NOT ALWAYS EXIST

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ABSTRACT. We construct a hyperbolic group with a hyperbolic subgroup for which inclusion does not induce a map of the boundaries.

2010 Mathematics Subject Classification: 20F67

Key words and phrases: hyperbolic group, Cannon–Thurston map

1. INTRODUCTION

Hyperbolic groups are the finitely generated groups whose Cayley graphs display characteristics of negative curvature. Their systematic study was initiated by Gromov in [?] and, mimicking the study of Riemannian manifolds, pays particular attention to large-scale and asymptotic geometric features such as boundaries.

One of the many equivalent definitions of the *Gromov boundary* ∂G of an infinite hyperbolic group G with word metric d is as the set of equivalence classes of sequences (a_n) in G such that

$$(a_m \cdot a_n)_e := (d(a_m, e) + d(a_n, e) - d(a_m, a_n))/2 \rightarrow \infty$$

as $m, n \rightarrow \infty$, where two such sequences (a_n) and (b_n) are equivalent when $(a_m \cdot b_n)_e \rightarrow \infty$ as $m, n \rightarrow \infty$. It is independent of the choice of finite generating set defining d and of the choice of basepoint. See [6] and [12] for surveys.

When H is an infinite hyperbolic subgroup of G , one can seek to induce a map $\partial H \rightarrow \partial G$ from the inclusion map. In [18] and [20] Mitra asks whether this is always well-defined, the concern being that ∂H is defined via the word metric on H and ∂G via that on G , and these may differ. He cites Bonahon [5] for similar questions and Bonahon [4], Floyd [8] and Minsky [16] for related work on Kleinian groups. The question is also raised by I. Kapovich & Benakli [12] and appears in the problem lists [3] and [13]. When the map exists, it is known as the Cannon–Thurston map.

The Cannon–Thurston map exists for many families of examples. The most straightforward is when H is quasi-convex (that is, undistorted). Cannon & Thurston [7] gave the first distorted example: they showed the map exists for $\pi_1 S \hookrightarrow \pi_1 M$ where M is a closed hyperbolic 3-manifold fibering over the circle with fiber a hyperbolic surface S (and, strikingly, the Cannon–Thurston map is a group-equivariant space-filling Peano curve $S^1 \twoheadrightarrow S^2$). Mitra generalized this widely. He showed the Cannon–Thurston map exists when H is an infinite normal subgroup of a hyperbolic group G [18], and he developed a theory of ending laminations for this context (inspired by [7]) to describe it [17]. He also showed

Date: March 27, 2013.

The second author gratefully acknowledges partial support from NSF grant DMS-1101651 and from Simons Foundation Collaboration Grant 208567.

the Cannon–Thurston map exists when H is one of the infinite vertex– or edge–groups of a finite graph of groups G in which G and all of the vertex– and edge–groups are hyperbolic, and all the defining monomorphisms from edge–groups to vertex–groups are quasi–isometric embeddings [19].

Nevertheless, we answer Mitra’s question negatively:

Theorem 1. *There are positive words C, C_1, C_2 on c_1, c_2 and $D_1, D_2, D_{11}, D_{12}, D_{21}, D_{22}$ on d_1, d_2 so that*

$$G = \left\langle a, b, c_1, c_2, d_1, d_2 \mid \begin{array}{ll} a^{-1}b^{-1}ab = C, & b^{-1}c_i b = C_i, \\ (ab)^{-1}d_j(ab) = D_j, & c_i^{-1}d_j c_i = D_{ij}, \quad 1 \leq i, j \leq 2 \end{array} \right\rangle$$

is hyperbolic, the subgroup

$$H = \langle b, d_1, d_2 \rangle$$

is free of rank 3, and there is no Cannon–Thurston map $\partial H \rightarrow \partial G$.

In fact, at the expense of complicating the construction, G can be made $\text{CAT}(-1)$, as we will outline in Remark 8.

That H is free is not pertinent to the nonexistence of the Cannon–Thurston map. Theorem 1 is the starting point for a proof by Matsuda and Oguni [15] that for every non–elementary hyperbolic group there is an embedding in some other hyperbolic group for which there is no Cannon–Thurston map.

Given that Cannon–Thurston maps do not always exist, the question arises as to what bearing subgroup distortion has. Heavy distortion appears to be no obstacle to the map’s existence: we showed in [1] that Cannon–Thurston maps exist for highly distorted free subgroups of hyperbolic hydra groups; these examples exhibit the maximum distortion known amongst hyperbolic subgroups of hyperbolic groups. As for small distortion, if a subgroup of a hyperbolic group is subexponentially distorted, then the subgroup is quasi–convex by Proposition 2.6 of [11] and so the Cannon–Thurston Map exists. The natural open question, then, (which Ilya Kapovich asked us) is whether there is an exponentially distorted hyperbolic subgroup of a hyperbolic group for which the Cannon–Thurston map does not exist.¹ It will be apparent from our proof of Theorem 1 that the subgroup $H \leq G$ is at least doubly–exponentially distorted.

Acknowledgement. We thank Ilya Kapovich and Hamish Short for comments and especially for Lemma 6, which replaces a more ad hoc argument in an earlier version of this article.

2. OUR CONSTRUCTION

We begin by showing that when C, C_i, D_j and D_{ij} are chosen suitably, the group G of Theorem 1 is hyperbolic.

A finite presentation for a group satisfies the $C'(\lambda)$ small–cancellation condition when, after cyclically reducing all the defining relations, the set S of all their cyclic permutations

¹An earlier version of this article claimed that the subgroup $G_{bcd} \leq G$ (defined before Lemma 4) is such an example. Although $\partial G_{bcd} \rightarrow \partial G$ is not well–defined, we realized that the distortion is at least doubly exponential, so Kapovich’s question remains open.

and those of their inverses, has the property that every common prefix between two distinct $r_1, r_2 \in S$ has length less than λ times the lengths of each of r_1 and r_2 [14, page 240].

Following Rips [21], we take

$$\begin{aligned} C &= c_1 c_2 c_1 c_2^2 c_1 c_2^3 \cdots c_1 c_2^r, \\ C_i &= c_1 c_2^{ri+1} c_1 c_2^{ri+2} c_1 c_2^{ri+3} \cdots c_1 c_2^{ri+r}, \\ D_j &= d_1 d_2^{rj+1} d_1 d_2^{rj+2} d_1 d_2^{rj+3} \cdots d_1 d_2^{rj+r}, \\ D_{ij} &= d_1 d_2^{r(il+j)+1} d_1 d_2^{r(il+j)+2} d_1 d_2^{r(il+j)+3} \cdots d_1 d_2^{r(il+j)+r}, \end{aligned}$$

where r is sufficiently large that the presentation for G of Theorem 1 satisfies the $C'(1/6)$ condition, and so G is hyperbolic. (All $C'(1/6)$ groups admit linear isoperimetric functions and so are hyperbolic [9].)

Next we analyze the construction of G to show (via Lemma 4 (iii), (iv)) that H is free of rank 3 for such C, C_i, D_j and D_{ij} .

Nielsen showed that a set of words represents a free basis for a subgroup of $F(X)$ when certain small-cancellation conditions are satisfied.

Proposition 2 (Nielsen, see [14, pages 6–7]). *A set U of words on an alphabet X represents a free basis for a subgroup of $F(X)$ if for every $v_1, v_2, v_3 \in U^{\pm 1}$,*

- N0. $v_1 \neq e$,
- N1. $v_1 v_2 \neq e \implies |v_1 v_2| \geq |v_1|, |v_2|$,
- N2. $v_1 v_2 \neq e$ and $v_2 v_3 \neq e \implies |v_1 v_2 v_3| > |v_1| - |v_2| + |v_3|$.

Corollary 3. *C, C_1, C_2 span a rank-3 free subgroup of $F(c_1, c_2)$ and $D_1, D_2, D_{11}, D_{12}, D_{21}, D_{22}$ span a rank-6 free subgroup of $F(d_1, d_2)$.*

(Indeed, N0–N2 are satisfied if U satisfies the $C'(1/2)$ property.)

Define

$$\begin{aligned} G_{cd} &:= \langle c_1, c_2, d_1, d_2 \mid c_i^{-1} d_j c_i = D_{ij}, \ 1 \leq i, j \leq 2 \rangle, \\ G_{bcd} &:= \langle G_{cd}, b \mid b^{-1} c_i b = C_i, \ 1 \leq i \leq 2 \rangle. \end{aligned}$$

Lemma 4. *The groups defined above have the following properties.*

- (i) G_{cd} is the fundamental group of a graph of groups consisting of a single vertex group $F(d_1, d_2)$ and a loop for each c_i . So both $F(c_1, c_2)$ and $F(d_1, d_2)$ are subgroups of G_{cd} and they have trivial intersection.
- (ii) G_{bcd} is an HNN-extension of G_{cd} with stable letter b mapping $c_i \mapsto C_i$.
- (iii) $H = \langle b, d_1, d_2 \rangle \leq G_{bcd}$ is free of rank 3.
- (iv) G of Theorem 1 is an HNN-extension of G_{bcd} with stable letter a :

$$G = \langle G_{bcd}, a \mid a^{-1} b a = b C^{-1}, \ a^{-1} d_j a = b D_j b^{-1}, \ 1 \leq j \leq 2 \rangle.$$

Proof. (i) This follows from the fact that for each i , the maps $F(d_1, d_2) \rightarrow F(d_1, d_2)$ given by $d_j \mapsto D_{ij}$ are injective, which is a consequence of Corollary 3.

(ii) Similarly, this follows from Corollary 3, which gives that the map $F(c_1, c_2) \rightarrow F(c_1, c_2)$ with $c_i \mapsto C_i$ is injective.

(iii) By Britton's Lemma's applied to the HNN-extension of (ii), any freely reduced word w on b, d_1, d_2 representing the identity would either contain no $b^{\pm 1}$ or contain a subword $b^{\pm 1}ub^{\mp 1}$ where u is a word on d_1, d_2 representing an element of $\langle c_1, c_2 \rangle$. In the latter case u must freely equal the identity as $F(c_1, c_2) \cap F(d_1, d_2) = \{e\}$ by (i). So w is the empty word as $\langle d_1, d_2 \rangle = F(d_1, d_2)$ by (i).

(iv) The presentation for G given arises from that in Theorem 1 by rewriting the defining relations involving a . We must show that $\langle bC^{-1}, bD_1b^{-1}, bD_2b^{-1} \rangle \leq G_{bcd}$ is free of rank 3. It suffices to show the same of the conjugate subgroup $\langle b^{-1}C, D_1, D_2 \rangle \leq G_{bcd}$, which we will do this by proving that if $i_1, \dots, i_{r-1} \neq 0$ and W_1, \dots, W_r are nontrivial elements of the rank-2 free group $F(D_1, D_2) \leq F(d_1, d_2) \leq G_{bcd}$, then

$$w := (b^{-1}C)^{i_0}W_1(b^{-1}C)^{i_1} \dots W_r(b^{-1}C)^{i_r}$$

does not represent the identity in G_{bcd} . This is achieved by showing as follows that it has nontrivial normal form when G_{bcd} is viewed as an HNN-extension as per (ii).

The relations $b^{-1}c_ib = C_i$ imply that $(b^{-1}C)^{i_k} \in \langle c_1, c_2 \rangle b^{-i_k} \langle c_1, c_2 \rangle$, so:

$$w \in \langle c_1, c_2 \rangle b^{-i_0} \langle c_1, c_2 \rangle W_1 \langle c_1, c_2 \rangle b^{-i_1} \langle c_1, c_2 \rangle \dots W_r \langle c_1, c_2 \rangle b^{-i_r} \langle c_1, c_2 \rangle.$$

If $b^{\pm 1}$ does not appear in w , then $r = 1, i_0 = i_1 = 0$, and $w = W_1$ does not represent the identity in G_{bcd} . So we may assume b appears. By Britton's Lemma it is enough to check that w has no subword $b^{\pm 1}Yb^{\mp 1}$ where Y is a word on c_1, c_2, d_1, d_2 representing an element of $\langle c_1, c_2 \rangle$. This is so because $F(c_1, c_2) \cap F(d_1, d_2) = \{1\}$ by (i) and $W_k \in F(d_1, d_2) \leq G_{bcd}$ does not represent the identity. \square

We will use the following lemma of Mitra to show the absence of a Cannon-Thurston map $\partial H \rightarrow \partial G$. We give our own account of this lemma in [1].

Lemma 5 (Mitra [18, 19]). *Suppose H is a hyperbolic subgroup of a hyperbolic group G and X_H and X_G are their Cayley graphs with respect to finite generating sets where that for H is a subset of that for G . (So X_H is a subgraph of X_G .) Let $M(N)$ be the infimal number such that if λ is a geodesic in X_H outside the ball of radius N about e in X_H , then every geodesic in X_G connecting the end-points of λ lies outside the ball of radius $M(N)$ about e in X_H . The Cannon-Thurston map $\partial H \rightarrow \partial G$ exists if and only if $M(N) \rightarrow \infty$ as $N \rightarrow \infty$.*

We will apply this to G and H of Theorem 1, using the generating sets a, b, c_1, c_2, d_1, d_2 and b, d_1, d_2 , respectively.

Our next lemma identifies some geodesics in Cayley graphs of small-cancellation groups. We learnt it from Ilya Kapovich and Hamish Short. It can be extracted from Strebel's appendix to [10] as we will explain. For a finite presentation $\langle A \mid R \rangle$, a word w on A is *Dehn-reduced* if every subword α of w that is a prefix of a cyclic conjugate of some $\rho \in R^{\pm 1}$ satisfies $|\alpha| \leq |\rho|/2$, and is *strongly Dehn-reduced* if $|\alpha| \leq |\rho|/6$.

Lemma 6. *If $\langle A \mid R \rangle$ is a $C'(1/6)$ -presentation, then every strongly Dehn-reduced word on A is geodesic. (Indeed, it is the unique geodesic word and also the unique Dehn-reduced word for the group element it represents.)*

Proof. Suppose u and v are freely reduced words which represent the same group element, and u is strongly Dehn-reduced and v is geodesic. In his proof of Proposition 39(i) in his

appendix to [10], Strebel explains that there is a van Kampen diagram Δ for uv^{-1} whose 2-dimensional portions are *ladder-like* disc-diagrams. (See the figure within Theorem 35.)

Suppose there is a 2-cell in Δ and that ρ is the defining relation one reads around its boundary. That 2-cell's boundary cycle is assembled from four paths: two run along the boundaries of adjacent 2-cells and have lengths less than $|\rho|/6$ (by the $C'(1/6)$ condition); one runs along u and has length at most $|\rho|/6$ by the strongly Dehn-reduced condition; but then the final path, which runs along v , has length more than $|\rho|/2$ contrary to v being a geodesic word. (Indeed, if we only required v to be Dehn-reduced we would get the same contradiction.) So Δ has no 2-cells and $u = v$ as words. \square

Proof of Theorem 1. We have seen that G , as presented in Theorem 1, is $C'(1/6)$ and so hyperbolic and that $H \leq G$ is free of rank 3 (Lemma 4 (iii), (iv)).

By Lemma 6, if r (introduced at the start of Section 2) is sufficiently large, then the word $b^{-n}a^{-n}d_1a^nb^n$ is *strongly* Dehn-reduced and so the path γ_n it labels, passing through the identity e as shown in Figure 1, is geodesic in the Cayley graph of G .

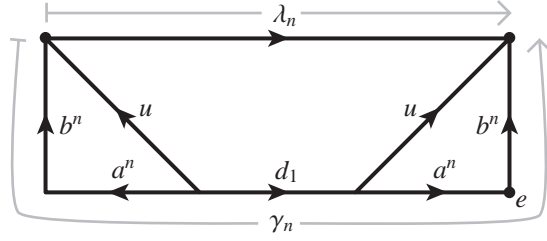


FIGURE 1. Paths in the Cayley graph of G illustrating our proof of Theorem 1.

The rewrite rules $ab \mapsto ba$ and $c_i b \mapsto b c_i$ are consequences of defining relations for G and can be used to convert $a^n b^n$ to a positive word u on ab, c_1, c_2 . So $b^{-n}a^{-n}d_1a^nb^n = u^{-1}d_1u$ in G , which equals a positive word on d_1, d_2 since $(ab)^{-1}d_j(ab) = D_j$ and $c_i^{-1}d_jc_i = D_{ij}$ in G . So the endpoints of γ_n are in H , and the geodesic λ_n joining them in the Cayley graph of H (which is a tree) is labelled by a word on d_1, d_2 . The distance (along the path labelled b^n) from e to λ_n in H is n .

As the distance from γ_n to e in the Cayley graph of G is zero and the distance from λ_n to e in the Cayley graph of H is n , there is no Cannon–Thurston map $\partial H \rightarrow \partial G$ by Lemma 5. \square

3. REMARKS

Remark 7. The inclusion $H \hookrightarrow G$ factors through G_{bcd} , which is also hyperbolic as its presentation is also $C'(1/6)$. So Theorem 1 implies the absence of at least one Cannon–Thurston map $\partial H \rightarrow \partial G_{bcd}$ or $\partial G_{bcd} \rightarrow \partial G$. In fact, more elaborate versions of the argument given above establish that both fail to exist. As an HNN-extension is an example of a graph of groups, the latter example also shows that the quasi-isometric embedding hypothesis in Mitra’s theorem from [19] is necessary.

Remark 8. With a similar construction, one can obtain a $\text{CAT}(-1)$ group G with a free subgroup H with no Cannon–Thurston map. Wise’s modification in [22] of the Rips construction [21] is used in [2] to construct $\text{CAT}(-1)$ groups. Each relator is realized on the

boundary of the unions of $n = 5$ congruent right-angled regular hyperbolic pentagons, arranged as row houses atop a geodesic segment. Each edge of the boundary corresponds to a generator. The vertices of the boundary are either right angles or straight angles, but the base geodesic gives $n - 1$ consecutive straight angles, bounding a segment of length $n - 2$. Wise shows that the Gromov link condition is satisfied when this straight segment is a freely reduced word and when the length- $(2n + 4)$ remainder of the boundary is obtained from the *Wise word*

$$(c_1 c_2 c_3 \cdots c_r)(c_2 c_3 \cdots c_r)(c_3 \cdots c_r) \cdots (c_r)$$

by chopping it into consecutive length $2n + 4$ segments (one for each defining relator). The argument works just as well for any n , so we take $n = 7$ and fit the $(ab)^{-1}d_j(ab)$, $a^{-1}b^{-1}ab$, $b^{-1}c_i b$, and $c_i^{-1}d_j c_i$ portions of our relators along the straight segment. We form one Wise word of c 's and one of d 's. To get sufficiently many length-18 subwords of the Wise words, we increase the number of c_i and d_j in the generating set for G . Then $H = \langle b, d_1, d_2, \dots \rangle$ is a free subgroup of the CAT(-1) group G by the same argument as before.

Remark 9. H has *infinite height* in G . That is, for all n , there exist $g_1, \dots, g_n \in G$ such that $\bigcap_{i=1}^n g_i^{-1} H g_i$ is infinite and $H g_i \neq H g_j$ for all $i \neq j$. Specifically, take $g_i = c_1^i$. Then, if $\phi : F(d_1, d_2) \rightarrow F(d_1, d_2)$ is the map $d_j \mapsto D_{1j}$ for $j = 1, 2$, then $\phi^n(F(d_1, d_2))$ is an infinite subgroup inside $g_i^{-1} H g_i$ for $1 \leq i \leq n$, and $H g_i \neq H g_j$ for all $i \neq j$ since $c_1^k \in H$ only for $k = 0$ by Lemma 4. Likewise, G_{bcd} has infinite height in G : instead of taking $g_i = c_1^i$, take $g_i = (ab)^i$ and apply the same argument as above. So our examples do not resolve the question attributed to Swarup in [20]: if H is a finitely presented subgroup of a hyperbolic group G of H has *finite height*, is H quasiconvex in G ?

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